Path-pependent parabolic PDEs and Path-dependent Feynman-Kac formula

Jocelyne Bion-Nadal
CNRS, CMAP Ecole Polytechnique

Bachelier Paris, january 8 2016

Dynamic Risk Measures and Path-Dependent second order PDEs, SEFE, Fred Benth and Giulia Di Nunno Eds, Springer Proceedings in Mathematics and Statistics Volume 138, 2016

OUTLINE

INTRODUCTION

The field of path dependent PDEs first started in 2010 when Peng asked in [Peng, ICM, 2010] wether a BSDE (Backward Stochastic Differential Equations) could be considered as a solution to a path dependent PDE.
In line with the recent litterature, a solution to a path dependent second order PDE

$$H(u,\omega,\phi(u,\omega),\partial_u\phi(u,\omega),D_x\phi(u,\omega),D_x^2\phi(u,\omega)) = 0 \quad (1)$$

is searched as a progressive function $\phi(u,\omega)$ (i.e. a path dependent function depending at time u on all the path ω up to time u).

CÀDLÀG PATHS

The notion of REGULAR SOLUTION for a path dependent PDE (1) needs to deal with càdlàg paths.
To define partial derivatives $D_x\phi(u,\omega)$ and $D_x^2\phi(u,\omega)$ at (u_0,ω_0), one needs to assume that $\phi(u_0,\omega)$ is defined for paths ω admitting a jump at time u_0.
S. Peng has introduced in [Peng 2012] a notion of regular and viscosity solution for a path dependent second order PDE based on the notions of continuity and partial derivatives introduced by Dupire [Dupire 2009].
The main drawback for this approach based on [Dupire 2009] is that the uniform norm topology on the set of càdlàg paths is not separable, it is not a Polish space.

Viscosity solution on continuous paths

Recently Ekren Keller Touzi and Zhang [2014] and also Ren Touzi Zhang [2014] proposed a notion of viscosity solution for path dependent PDEs in the setting of continuous paths. These works are motivated by the fact that a continuous function defined on the set of continuous paths does not have a unique extension into a continuous function on the set of càdlàg paths.

NEW APPROACH

In the paper [Dynamic Risk Measures and Path-Dependent second order PDEs,2015] I introduce a new notion of regular and viscosity solution for path dependent second order PDEs, making use of the Skorokhod topology on the set of càdlàg paths. Thus Ω is a Polish space. To define the regularity properties of a progressive function ϕ we introduce a one to one correspondance between progressive functions in 2 variables and strictly progressive functions in 3 variables.
Our study allows then to define the notion of viscosity solution for path dependent functions defined only on the set of continuous paths.

Construction of Solutions

Making use of the Martingale Problem Approach for integro differential operators with path dependent coefficients [J. Bion-Nadal 2015], we construct then time-consistent dynamic risk measures on the set Ω of càdlàg paths. These risk measures provide viscosity solutions for path dependent semi-linear second order PDEs.
This approach is motivated by the Feynman Kac formula and more specifically by the link between solutions of parabolic second order PDEs and probability measures solutions to a martingale problem. The martingale problem has been first introduced and studied by Stroock and Varadhan (1969) in the case of continuous diffusion processes.

OUTLINE

TOPOLOGY

In all the following, Ω IS THE SET OF CÀDLÀG PATHS $\mathcal{D}(\boldsymbol{R}_+, \boldsymbol{R}^n)$ ENDOWED WITH THE SKOROKHOD TOPOLOGY
$d(\omega_n, \omega) \to 0$ if there is a sequence $\lambda_n : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ strictly increasing, $\lambda_n(0) = 0$, such that $||\mathrm{Id} - \lambda_n||_\infty \to 0$, and for all $K > 0$,
$\sup_{t \leq K} ||\omega(t) - \omega_n \circ \lambda_n(t)|| \to 0$

THE SET OF CÀDLÀG PATHS WITH THE SKOROKHOD TOPOLOGY IS A POLISH SPACE (metrizable and separable). Polish spaces have nice properties:

- Existence of regular conditional probability distributions
- Equivalence between relative compactness and tightness for a set of probability measures
- The Borel σ-algebra is countably generated.

THE SET OF CÀDLÀG PATHS WITH THE UNIFORM NORM TOPOLOGY IS NOT A POLISH SPACE. It is not separable.

NEW APPROACH FOR PROGRESSIVE FUNCTIONS

DEFINITION

Let Y be a metrizable space. A function $f : \boldsymbol{R}_+ \times \Omega \rightarrow Y$ is progressive if $f(s,\omega) = f(s,\omega')$ for all ω, ω' such that $\omega_{|[0,s]} = \omega'_{|[0,s]}$.

To every progressive function $f : \boldsymbol{R}_+ \times \Omega \rightarrow Y$ we associate a unique function $\bar{f}$ defined on $\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n$ by

$$\bar{f}(s,\omega,x) = f(s,\omega *_s x)$$

$$\begin{aligned} \omega *_s x(u) &= \omega(u) \ \ \forall u < s \\ \omega *_s x(u) &= x \ \ \forall s \leq u \end{aligned} \qquad (2)$$

$\bar{f}$ is strictly progressive, i.e. $\bar{f}(s,\omega,x) = \bar{f}(s,\omega',x)$ if $\omega_{|[0,s[} = \omega'_{|[0,s[}$

$f \rightarrow \bar{f}$ is a one to one correspondance, $f(s,\omega) = \bar{f}(s,\omega,X_s(\omega))$.

REGULAR SOLUTION OF A PATH DEPENDENT PDE

DEFINITION

A progressive function v on $\boldsymbol{R}_+ \times \Omega$ is a regular solution to the following path dependent second order PDE

$$H(u,\omega,v(u,\omega),\partial_u v(u,\omega),D_x v(u,\omega),D_x^2 v(u,\omega)) = 0 \quad (3)$$

if the function $\overline{v}$ belongs to $\mathcal{C}^{1,0,2}(\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n)$ and if the usual partial derivatives of $\overline{v}$ satisfy the equation

$$H(u,\omega *_u x,\overline{v}(u,\omega,x),\partial_u \overline{v}(u,\omega,x),D_x \overline{v}(u,\omega,x),D_x^2 \overline{v}(u,\omega,x) = 0 \quad (4)$$

with $\overline{v}(u,\omega,x) = v(u,\omega *_u x)$
$(\omega *_u x)(s) = \omega(s)\ \forall s < u$, and $(\omega *_u x)(s) = x\ \forall s \geq u$. The partial derivatives of $\overline{v}$ are the usual one, the continuity notion for $\overline{v}$ is the usual one.

Sufficient to assume that $\overline{v} \in \mathcal{C}^{1,0,2}(X)$ where $X = \{(s,\omega,x),\ \omega = \omega *_s x\}$

Continuity in viscosity sense

Definition

A progressively measurable function v defined on $\boldsymbol{R}_+ \times \Omega$ is continuous in viscosity sense at (r, ω_0) if

$$v(r, \omega_0) = \lim_{\epsilon \longrightarrow 0} \{v(s, \omega),\ (s, \omega) \in D_\epsilon(r, \omega_0)\} \tag{5}$$

where

$$\begin{aligned} D_\epsilon(r, \omega_0) = \quad & \{(s, \omega),\ r \leq s < r + \epsilon,\ \omega(u) = \omega_0(u),\ \forall 0 \leq u \leq r \\ & \omega(u) = \omega(s)\ \forall u \geq s,\ \text{and}\ \sup\nolimits_{r \leq u \leq s} ||\omega(u) - \omega_0(r)|| < \epsilon\} \end{aligned} \tag{6}$$

v is lower (resp upper) semi continuous in viscosity sense if equation (5) is satisfied replacing lim by lim inf(resp lim sup).

VISCOSITY SUPERSOLUTION ON THE SET OF CÀDLÀG PATHS

DEFINITION

Let v be a progressively measurable function on $(\boldsymbol{R}_+ \times \Omega, (\mathcal{B}_t))$ where Ω is the set of càdlàg paths with the Skorokhod topology and $(\mathcal{B}_t)$ the canonical filtration.

v is a viscosity supersolution of (3) if v is lower semi-continuous in viscosity sense, and if for all $(t_0, \omega_0) \in \boldsymbol{R}_+ \times \Omega$, there exists $\epsilon > 0$ such that

- v is bounded from below on $D_\epsilon(t_0, \omega_0)$.
- for all strictly progressive function $\overline{\phi} \in \mathcal{C}_b^{1,0,2}(\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n)$ such that $v(t_0, \omega_0) = \overline{\phi}(t_0, \omega_0, \omega_0(t_0))$, and (t_0, ω_0) is a minimizer of $v - \phi$ on $D_\epsilon(t_0, \omega_0)$.

$$H(u, \omega *_u x, \overline{\phi}(u, \omega, x), \partial_u \overline{\phi}(u, \omega, x), D_x \overline{\phi}(u, \omega, x), D_x^2 \overline{\phi}(u, \omega, x) \geq 0$$

at point $(t_0, \omega_0, \omega_0(t_0))$.

Viscosity solution on continuous paths

Definition

A progressively measurable function v on $\boldsymbol{R}_+ \times \mathcal{C}(\boldsymbol{R}_+, \boldsymbol{R}^n)$ is a viscosity supersolution of $H(u, \omega, v(u, \omega), \partial_u v(u, \omega), D_x v(u, \omega), D_x^2 v(u, \omega)) = 0$ if v satisfies the conditions of the previous theorem replacing $D_\epsilon(r, \omega_0)$ by $\tilde{D}_\epsilon(r, \omega_0)$ where $\tilde{D}_\epsilon(r, \omega_0)$ is the intersection of $D_\epsilon(r, \omega_0)$ with the set of continuous paths

OUTLINE

MARTINGALE PROBLEM OF STROOCK AND VARADHAN

The martingale problem associated with a second order elliptic differential operator has been introduced and studied By Stroock and Varadhan ["Diffusion processes with continuous coefficients I and II", Communications on Pure and Applied Mathematics,1969]
Second order elliptic differential operator:

$$L_t^{a,b} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x) \frac{\partial}{\partial x_i}$$

The operator $L^{a,b}$ is acting on $\mathcal{C}_0^\infty(\boldsymbol{R}^n)$ (functions $\mathcal{C}^\infty$ with compact support).

MARTINGALE PROBLEM OF STROOCK AND VARADHAN

State space: $(\mathcal{C}([0,\infty[,\boldsymbol{R}^n)$; X_t is the canonical process: $X_t(\omega)=\omega(t)$
$\mathcal{B}_t$ is the σ-algebra generated by $(X_u)_{u\leq t}$.
Let $0\leq r$ and $y\in\boldsymbol{R}^n$. A PROBABILITY MEASURE Q on the space of continuous paths $\mathcal{C}([0,\infty[,\boldsymbol{R}^n)$ IS A SOLUTION TO THE MARTINGALE PROBLEM FOR $L^{a,b}$ starting from y at time r if for all $f\in\mathcal{C}_0^\infty(\boldsymbol{R}^n)$,

$$Y^{a,b}_{r,t}=f(X_t)-f(X_r)-\int_r^t L^{a,b}_u(f)(u,X_u)du \tag{7}$$

is a Q martingale on $(\mathcal{C}([0,\infty[,\boldsymbol{R}^n),\mathcal{B}_t)$ and if $Q(\{\omega(u)=y\ \ \forall u\leq r\})=1$

$$L^{a,b}_u(f)(u,X_u)=\frac{1}{2}\sum_1^n a_{ij}(u,X_u)\frac{\partial^2 f}{\partial x_i\partial x_j}(X_u)+\sum_1^n b_i(u,X_u)\frac{\partial f}{\partial x_i}(X_u)$$

FEYNMAN KAC FORMULA

Stroock and Varadhan have proved the existence and the uniqueness of the solution to the martingale problem associated to the operator $L^{a,b}$ starting from x at time t assuming that a is a continuous bounded function on $\boldsymbol{R}_+ \times \boldsymbol{R}^n$ with values in the set of definite positive matrices and b is measurable bounded: $Q^{a,b}_{t,x}$

THE FEYNMAN KAC FORMULA establishes a link between a solution of a parabolic second order PDE and probability measures solutions to a martingale problem. Under regularity conditions there is a unique solution v to the PDE: $\partial_u v(t,x) + \mathcal{L}^{a,b} v(t,x) = 0$, $v(T,.) = h$ with $\mathcal{L}^{a,b} v(t,x) = \frac{1}{2} Tr(a(t,x)) D^2_x(v)(t,x) + b(t,x)^* D_x v(t,x)$.

From the FEYNMAN KAC FORMULA $v(t,x) = E_{Q^{a,b}_{t,x}}(h(X_T))$

(X_u) is the canonical process.

One natural way to construct and study path dependent parabolic second order partial differential equations is thus to start with probability measures solution to the path dependent martingale problem associated to the operator $L^{a,b}$ for path dependent coefficients a and b.

PATH DEPENDENT MARTINGALE PROBLEM

I have recently studied the martingale problem associated with an integro differential operator with path dependent coefficients. We consider here the case where there is no jump term.
We consider the following path dependent operator:

$$L^{a,b}(t,\omega) = \frac{1}{2}\sum_1^n a_{ij}(t,\omega)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_1^n b_i(t,\omega)\frac{\partial}{\partial x_i} \quad (8)$$

The functions a and b are defined on $\boldsymbol{R}_+ \times \Omega$ where Ω is the set of càdlàg paths. For given t, $a(t,\omega)$ and $b(t,\omega)$ depend on the whole trajectory of ω up to time t.

Path dependent martingale problem

Let $\Omega = \mathcal{D}([0, \infty[, \boldsymbol{R}^n)$.

Definition

Let $r \geq 0$, $\omega_0 \in \Omega$. A probability measure Q on the space Ω is a solution to the path dependent martingale problem for $L^{a,b}(t, \omega)$ starting from ω_0 at time r if for all $f \in \mathcal{C}_0^\infty(\boldsymbol{R}^n)$,

$$Y_{r,t}^{a,b,M} = f(X_t) - f(X_r) - \int_r^t (L^{a,b}(u, \omega)(f)(X_u)du \tag{9}$$

is a Q martingale on $(\Omega, \mathcal{B}_t)$ and if

$$Q(\{\omega \in \Omega \; |\omega_{|[0,r]} = \omega_{0|[0,r]}\}) = 1$$

PATH DEPENDENT MARTINGALE PROBLEM

THEOREM

Assume that a and b are bounded. Let Q be a probability measure on Ω such that $Q(\{\omega \in \Omega \,|\omega_{|[0,r]} = \omega_{0|[0,r]}\}) = 1$.
The following properties are equivalent :

- *For all $f \in \mathcal{C}_0^\infty(\mathbf{R}^d)$,*

$$Y_{r,t}^{a,b,M}(f) = f(X_t) - f(X_r) - \int_r^t L^{a,b}(u,\omega)(f)(X_u)du \tag{10}$$

  is a $(Q,\mathcal{B}_t)$ martingale

- *For all $f \in \mathcal{C}_b^{1,2}(\mathbf{R}_+ \times \mathbf{R}^n)$, $Z_{r,t}^{a,b,M}(f) =$*

$$f(t,X_t) - f(r,X_r) - \int_r^t (\frac{\partial}{\partial u} + L^{a,b}(u,\omega)(f)(u,X_u)du \tag{11}$$

  is a $(Q,\mathcal{B}_t)$ martingale.

PATH DEPENDENT MARTINGALE PROBLEM

THEOREM

- *For all* $\phi \in \mathcal{C}_b^{1,0,2}(\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n)$ *strictly progressive,*

$$\phi(t,\omega,X_t) - \phi(r,\omega,X_r) - \int_r^t [\frac{\partial}{\partial u} + L^{a,b}(u,\omega)]\phi(u,\omega,X_u(\omega))du$$

  is a $(Q,\mathcal{B}_t)$ *martingale.*

- *For all* $g : \boldsymbol{R}_+ \times \Omega \rightarrow \boldsymbol{R}$ *progressive, such that* $\overline{g}$ $(\overline{g}(s,\omega,x) = g(s,\omega *_s x))$ *belongs to* $\mathcal{C}_b^{1,0,2}(\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n)$,

$$g(t,\omega) - g(r,\omega) - \int_r^t [\frac{\partial}{\partial u} + L^{a,b}(u,\omega)](\overline{g})(u,\omega,X_u(\omega))du$$

  is a $(Q,\mathcal{B}_t)$ *martingale.*

PATH DEPENDENT MARTINGALE PROBLEM

For $\phi \in \mathcal{C}_b^{1,0,2}(\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n)$,

$$L^{a,b}(u,\omega)(\phi)(u,\omega,X_u(\omega)) =$$

$$+\frac{1}{2}\sum_1^n a_{ij}(u,\omega)\frac{\partial^2\phi}{\partial x_i \partial x_j}(u,\omega,X_u(\omega)) + \sum_1^n b_i(u,\omega)\frac{\partial\phi}{\partial x_i}(u,\omega,X_u(\omega))$$

The martingale problem studied by Stroock and Varadhan is a particular case of the above path dependent martingale problem with $a(t,\omega) = \tilde{a}(t,X_t(\omega))$, $b(t,\omega) = \tilde{b}(t,X_t(\omega))$, $\tilde{a}$,$\tilde{b}$ defined on $\boldsymbol{R}_+ \times \boldsymbol{R}^n$. WHICH CONTINUITY ASSUMPTION ON a? Recall that Ω is the set of càdlàg paths.

DEFINITION

A progressive function ϕ defined on $\boldsymbol{R}_+ \times \Omega$ is progressively continuous if $\overline{\phi}$ ($\overline{\phi}(u,\omega,x) = \phi(u,\omega *_u x)$) is continuous on $\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n$.

Motivation: If $\tilde{a}$ is continuous, a given by $a(t,\omega) = \tilde{a}(t,X_t(\omega))$ is progressively continuous but not continuous on the set of càdlàg paths.

Existence and uniqueness of a solution

Theorem

Let a be a progressively continuous bounded function defined on $\mathbf{R}_+ \times \Omega$ with values in the set of non negative matrices.
Assume that $a(s,\omega)$ is invertible for all (s,ω).
Let b be a progressively measurable bounded function defined on $\mathbf{R}_+ \times \Omega$ with values in $\mathbf{R}^n$.
For all (r,ω_0), the martingale problem for $\mathcal{L}^{a,ab}$ starting from ω_0 at time r admits a solution $Q^{a,ab}_{r,\omega_0}$ on the set of càdlàg paths.
Under a stronger continuity assumption on a (including the Lipschitz case) there is a unique solution to the martingale problem for $\mathcal{L}^{a,ab}$ starting from ω_0 at time r.
If the function a is δ-delayed which means that $a(u,\omega) = \overline{a}(u,\omega,X_u(\omega))$ where $\overline{a}(u,\omega,x)$ depends only on ω up to time $u-\delta$ there is also a unique solution to the martingale problem for $\mathcal{L}^{a,ab}$ starting from ω_0 at time r.

THE ROLE OF CONTINUOUS PATHS

PROPOSITION

Every probability measure $Q^{a,ab}_{r,\omega_0}$ solution to the martingale problem for $\mathcal{L}^{a,ab}$ starting from ω_0 at time r is supported by paths which are continuous after time r, i.e.continuous on $[r,\infty[$.
More precisely

$$Q^{a,ab}_{r,\omega_0}(\{\omega,\ \omega(u)=\omega_0(u)\ \forall u\leq r,\ \text{and}\ \omega_{|[r,\infty[}\in\mathcal{C}([r,\infty[,\boldsymbol{R}^n)\}=1$$

COROLLARY

For all continuous path ω_0 and all r, the support of the probability measure $Q^{a,ab}_{r,\omega_0}$ is contained in the set of continuous paths:

$$Q^{a,b}_{r,\omega_0}\mathcal{C}([\boldsymbol{R}_+,\boldsymbol{R}^n))=1$$

Feller property

Theorem

Assume that a and b are progressively continuous bounded. Assume that there is a unique solution to the martingale problem for $\mathcal{L}^{a,0}$ starting from ω_0 at time r. Then there is a unique solution to the martingale problem for $\mathcal{L}^{a,ab}$ starting from ω_0 at time r. Furthermore the map

$$(r,\omega,x) \in \boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n \to Q^{a,ab}_{r,\omega *_r x} \in \mathcal{M}_1(\Omega)$$

is continuous on $\{(r,\omega,x) \mid \omega = \omega *_r x\}$
The set of probability measures $\mathcal{M}_1(\Omega)$ is endowed with the weak topology.

Let $h(\omega) = \overline{h}(\omega, \omega(T))$, $\overline{h}$ continuous, $\overline{h}(\omega,x) = \overline{h}(\omega',x)$ if $\omega(u) = \omega'(u),\ \forall u < T$.

Proposition

Let $v(r,\omega) = Q^{a,ab}_{r,\omega}(h)$. $\bar{v}$ is continuous on X. The function v is a viscosity solution of $\partial_t v(t,\omega) + L^{a,ab} v(t,\omega) = 0\,,\ v(T,\omega) = h(\omega)$

PATH DEPENDENT INTEGRO DIFFERENTIAL OPERATORS

Consider the following path dependent operators

$$L^{a,b}(t,\omega) = \frac{1}{2}\sum_{1}^{n} a_{ij}(t,\omega)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{1}^{n} b_i(t,\omega)\frac{\partial}{\partial x_i} \quad (12)$$

and

$$K^M(t,\omega)(f)(x) = \int_{R^n-\{0\}} [f(x+y) - f(x) - \frac{y^*\nabla f(x)}{1+||y||^2}]M(t,\omega,dy) \quad (13)$$

a, b, M are progressive functions.

PATH DEPENDENT MARTINGALE PROBLEM

Let $\Omega = \mathcal{D}([0,\infty[, \boldsymbol{R}^n)$.

DEFINITION

Let $r \geq 0$, $\omega_0 \in \Omega$. A probability measure Q on the space Ω is a solution to the path dependent martingale problem for $L^{a,b}(t,\omega) + K^M(t,\omega)$ starting from ω_0 at time r if for all $f \in \mathcal{C}_0^\infty(\boldsymbol{R}^n)$,

$$Y_{r,t}^{a,b,M} = f(X_t) - f(X_r) - \int_r^t (L^{a,b}(u,\omega) + K^M(u,\omega))(f)(X_u)du \qquad (14)$$

is a Q martingale on $(\Omega, \mathcal{B}_t)$ and if

$$Q(\{\omega \in \Omega \; |\omega_{|[0,r]} = \omega_{0|[0,r]}\}) = 1$$

(A)-BOUNDEDNESS CONDITION

DEFINITION

The functions a, b, M satisfy the (A)-boundedness condition if

$$\sup_{s\geq 0,\omega\in\Omega} ||a(s,\omega)|| \leq A$$

$$\sup_{s\geq 0,\omega\in\Omega} ||b(s,\omega)|| \leq A$$

$$\sup_{s\geq 0,\omega\in\Omega} \int_{R^n-\{0\}} \frac{||y||^2}{1+||y||^2} M(s,\omega,dy) \leq A$$

EXISTENCE AND UNIQUENESS

THEOREM

Assume that

- (a, b, M) *satisfy the A-boundedness condtion*
- $\forall s, \omega,\ a(s, \omega)$ *is positive definite*
- *a is progressively continuous (i.e.* $(s, \omega, x) \rightarrow \overline{a}(s, \omega, x) = a(s, \omega *_s x)$ *is continuous)*
- $\forall \phi, \quad (s, \omega, x) \rightarrow \int_{R^d - \{0\}} \frac{||y||^2}{1+||y||^2} \phi(y) M(s, \omega *_s x, dy)$ *is continuous.*

Then for all (r, ω_0) *there exists a solution* Q_{r,ω_0} *to the path dependent martingale problem for* $L^{a,b}(t, \omega) + K^M(t, \omega)$ *starting from* ω_0 *at time r, i.e. such that*

$$Q_{r,\omega_0}(\{\omega |\ \omega_{|[0,r]} = \omega_{0|[0,r]}\}) = 1$$

Under more restrictive assumptions the solution is unique.

Outline

TIME CONSISTENT DYNAMIC RISK MEASURES

Recall the following way of constructing time consistent dynamic risk measures [J.Bion-Nadal 2008].

PROPOSITION

Given a stable set $\mathcal{Q}$ of probability measures all equivalent to Q_0 and a penalty $(\alpha_{s,t})$ defined on $\mathcal{Q}$ satisfying the local property and the cocycle condition,

$$\rho_{st}(X) = \mathrm{esssup}_{Q \in \mathcal{Q}}(E_Q(X|\mathcal{F}_s) - \alpha_{st}(Q))$$

defines a time consistent dynamic risk measure on $L_\infty(\Omega, \mathcal{B}, (\mathcal{B}_t), Q_0)$ or on $L_p(\Omega, \mathcal{B}, (\mathcal{B}_t), Q_0)$ if the corresponding integrability conditions are satisfied. That is: $\rho_{st} : L_p(\Omega, \mathcal{B}_t, Q_0) \to L_p(\Omega, \mathcal{B}_s, Q_0)$, ρ_{st} is convex, translation invariant by elements of $L_p(\Omega, \mathcal{B}_s, Q_0)$ and $\rho_{r,t}(X) = \rho_{r,s}(\rho_{s,t}(X))$ for all $X \in L_p(\Omega, \mathcal{B}_t, Q_0)$ and $r \leq s \leq t$.

Stable set of propbability measures

Definition

A set $\mathcal{Q}$ of equivalent probability measures on a filtered probability space $(\Omega, \mathcal{B}, (\mathcal{B}_t))$ is stable if it satisfies the two following properties:

1. Stability by composition
   For all $s \geq 0$ for all Q and R in $\mathcal{Q}$, there is a probability measure S in $\mathcal{Q}$ such that for all X bounded $\mathcal{B}$-measurable,
   $$E_S(X) = E_Q(E_R(X|\mathcal{B}_s))$$
2. Stability by bifurcation
   For all $s \geq 0$, for all Q and R in $\mathcal{Q}$, for all $A \in \mathcal{B}_s$, there is a probability measure S in $\mathcal{Q}$ such that for all X bounded $\mathcal{B}$-measurable,
   $$E_S(X|\mathcal{B}_s) = 1_A E_Q(X|\mathcal{B}_s) + 1_{A^c} E_R(X|\mathcal{B}_s)$$

PENALTIES

DEFINITION

A penalty function α defined on a stable set $\mathcal{Q}$ of probability measures all equivalent is a family of maps $(\alpha_{s,t}),\ s \leq t$, defined on $\mathcal{Q}$ with values in the set of $\mathcal{B}_s$-measurable maps.

I) It is local:
if for all Q, R in $\mathcal{Q}$, for all s, for all A in $\mathcal{B}_s$, the assertion $1_A E_Q(X|\mathcal{B}_s) = 1_A E_R(X|\mathcal{B}_s)$ for all X bounded $\mathcal{B}_t$ measurable implies that $1_A \alpha_{s,t}(Q) = 1_A \alpha_{s,t}(R)$.

II) It satisfies the cocycle condition if for all $r \leq s \leq t$, for all Q in $\mathcal{Q}$,

$$\alpha_{r,t}(Q) = \alpha_{r,s}(Q) + E_Q(\alpha_{s,t}(Q)|\mathcal{F}_r)$$

Multivalued mapping

For all (s,ω,x) consider $\Lambda(s,\omega,x) \subset \boldsymbol{R}^n$ such that $\Lambda(s,\omega,x) = \Lambda(s,\omega',x)$ if $\omega(u) = \omega'(u)\ \forall u \leq s$. Λ is a multivalued mapping on X (the quotient of $\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n$ by the equivalence relation $\sim$: $(t,\omega,x) \sim (t',\omega',x')$ if $t = t'$, $x = x'$ and $\omega(u) = \omega'(u)\ \forall u < t$).
A selector from Λ is a map s such that $s(t,\omega,x) \in \Lambda(t,\omega,x)$ for all (t,ω,x).

Definition

A multivalued mapping Λ from X into $\boldsymbol{R}^n$ is lower hemicontinuous if for every closed subset F of $\boldsymbol{R}^n$, $\{(t,\omega,x) \in X : \Lambda(t,\omega,x) \subset F\}$ is closed.

Recall the following Michael selection Theorem:
A lower hemicontinuous mapping from a metrizable space into a Banach space with non empty closed convex values admits a continuous selector.

STABLE SET OF PROBABILITY MEASURES SOLUTION TO A MARTINGALE PROBLEM

In all the following a is progressively continuous bounded on $\boldsymbol{R}_+ \times \Omega$ $a(s,\omega)$ invertible for all (s,ω) . For all (r,ω) there is a unique solution $Q^{a,0}_{r,\omega}$ to the martingale problem for $\mathcal{L}^{a,0}$ starting from ω at time r.

DEFINITION

Let Λ be a closed convex lower hemicontinuous multivalued mapping. Let $L(\Lambda)$ be the set of continuous bounded selectors from Λ.
Let $r \geq 0$ and $\omega \in \Omega$. The set $\mathcal{Q}_{r,\omega}(\Lambda)$ is the stable set of probability measures generated by the probability measures $Q^{a,a\lambda}_{r,\omega}, \overline{\lambda} \in L(\Lambda)$ with $\lambda(t,\omega') = \overline{\lambda}(t,\omega',X_t(\omega'))$

Let $\mathcal{P}$ be the predictable σ-algebra. Every probability measure in $\mathcal{Q}_{r,\omega}(\Lambda)$ is the unique solution $Q^{a,a\mu}_{r,\omega}$ to the martingale problem for $L^{a,a\mu}$ starting from ω at time r for some process μ defined on $\boldsymbol{R}_+ \times \Omega \times \boldsymbol{R}^n$ $\mathcal{P} \times \mathcal{B}(\boldsymbol{R}^n)$-measurable Λ valued $(\mu(u,\omega,x) \in \Lambda(u,\omega,x))$.
Every probability measure in $\mathcal{Q}_{r,\omega}(\Lambda)$ is equivalent with $Q^{a}_{r,\omega} = Q^{a,0}_{r,\omega}$.

PENALTIES

For $0 \leq r \leq s \leq t$, define the penalty $\alpha_{s,t}(Q^{a,a\mu}_{r,\omega})$ as follows

$$\alpha_{s,t}(Q^{a,a\mu}_{r,\omega}) = E_{Q^{a,a\mu}_{r,\omega}}(\int_s^t g(u,\omega,\mu(u,\omega))du|\mathcal{B}_s) \tag{15}$$

where g is $\mathcal{P} \times \mathcal{B}(\boldsymbol{R}^n)$-measurable.

Growth conditions

Definition

1. g satisfies the growth condition (GC1) if there is $K > 0$, $m \in N^*$ and $\epsilon > 0$ such that

$$\forall y \in \Lambda(u,\omega,X_u(\omega)),\ |g(u,\omega,y)| \leq K(1+\sup_{s\leq u}||X_s(\omega)||)^m(1+||y||^{2-\epsilon}) \tag{16}$$

2. g satisfies the growth condition (GC2) if there is $K > 0$ such that

$$\forall y \in \Lambda(u,\omega,X_u(\omega)),\ |g(u,\omega,y)| \leq K(1+||y||^2) \tag{17}$$

BMO CONDITION

DEFINITION

Let $C > 0$. Let Q be a probability measure.

- A progressively measurable process μ belongs to $BMO(Q)$ and has a BMO norm less or equal to C if for all stopping times τ,

$$E_Q(\int_\tau^\infty ||\mu_s||^2 ds|\mathcal{F}_\tau) \leq C$$

- The multivalued mapping Λ is $BMO(Q)$ if there is a map $\phi \in BMO(Q)$ such that

$$\forall (u,\omega),\ \sup\{||y||,\ y \in \Lambda(u,\omega)\} \leq \phi(u,\omega)$$

Time consistent dynamic risk measure on L_p

Theorem

Let (r, ω). Assume that the multivalued set Λ is BMO$(Q^a_{r,\omega})$. Let $\mathcal{Q} = \mathcal{Q}_{r,\omega}(\Lambda)$ Let $r \leq s \leq t$.

$$\rho^{r,\omega}_{s,t}(Y) = \text{esssup}_{\mathrm{Q}^{\mathrm{a},\mathrm{a}\mu}_{\mathrm{r},\omega} \in \mathcal{Q}}(\mathrm{E}_{\mathrm{Q}^{\mathrm{a},\mathrm{a}\mu}_{\mathrm{r},\omega}}(\mathrm{Y}|\mathcal{B}_{\mathrm{s}}) - \alpha_{\mathrm{s},\mathrm{t}}(\mathrm{Q}^{\mathrm{a},\mathrm{a}\mu}_{\mathrm{r},\omega}))$$

with $\alpha_{s,t}(Q^{a,a\mu}_{r,\omega}) = E_{Q^{a,a\mu}_{r,\omega}}(\int_s^t g(u, \omega, \mu(u, \omega))du|\mathcal{B}_s)$

- *Assume that g satisfies the growth condition (GC1). Then $(\rho^{r,\omega}_{s,t})$ defines a time consistent dynamic risk measure on $L_p(Q^a_{r,\omega}, (\mathcal{B}_t))$ for all $q_0 \leq p < \infty$.*
- *Assume that g satisfies the growth condition (GC2). Then $(\rho^{r,\omega}_{s,t})$ defines a time consistent dynamic risk measure on $L_p(Q^a_{r,\omega}, (\mathcal{B}_t))$ for all $q_0 \leq p \leq \infty$*

q_0 is linked to the BMO norm of the majorant of Λ.

Feller property for the dynamic risk measure

Definition

The function $h : \Omega \to \mathbf{R}$ belongs to $\mathcal{C}_t$ if there is a $\tilde{h} : \Omega \times \mathbf{R}^n \to \mathbf{R}$ such that

- $h(\omega) = \tilde{h}(\omega, X_t(\omega))$
- $\tilde{h}(\omega, x) = \tilde{h}(\omega', x)$ if $\omega(u) = \omega'(u) \ \ \forall u < t$

and such that $\tilde{h}$ is continuous bounded on $\{(\omega, x), \ \omega = \omega *_t x\} \subset \Omega \times \mathbf{R}^n$

Theorem

Under the same hypothesis. For every function $h \in \mathcal{C}_t$, there is a progressive map $R(h)$ on $\mathbf{R}_+ \times \Omega$, $R(h)(t, \omega) = h(\omega)$, such that $\overline{R}(h)$ is lower semi continuous on $\{(u, \omega, x), \ \omega = \omega *_u x, \ u \leq t\}$.

$$\forall s \in [r, t], \ \forall \omega' \in \Omega, \ \rho_{s,t}^{s,\omega'}(h) = R(h)(s, \omega') \tag{18}$$

$$\forall 0 \leq r \leq s \leq t, \ \ \rho_{s,t}^{r,\omega}(h)(\omega') = R(h)(s, \omega', \omega'(s)) \ \ Q^a_{r,\omega} \ a.s. \tag{19}$$

VISCOSITY SOLUTION

THEOREM

Assume furthermore that g is upper semicontinuous on $\{(s,\omega,y),(s,\omega)\in X,\ y\in\Lambda(s,\omega)\}$. Let $h\in\mathcal{C}_t$. The function $R(h)$ is a viscosity supersolution of the path dependent second order PDE

$$\begin{aligned} -\partial_u v(u,\omega) - \mathcal{L}v(u,\omega) - f(u,\omega,a(u,\omega)D_x v(u,\omega)) &= 0 \\ v(t,\omega) &= f(\omega) \end{aligned}$$

$$\begin{aligned} \mathcal{L}v(u,\omega) &= \frac{1}{2}Tr(a(u,\omega)D_x^2(v)(u,\omega)) \\ f(u,\omega,z) &= \sup_{y\in\Lambda(u,\omega)} (z^*y - g(u,\omega,y)) \end{aligned}$$

at each point (t_0,ω_0) such that $f(t_0,\omega_0,a(t_0,\omega_0)z)$ is finite for all z.

$\rho_{s,t}^{s,\omega'}(h) = R(h)(s,\omega')$

VISCOSITY SOLUTION

THEOREM

Assume furthermore that Λ is uniformly BMO with respect to a. Assume that f is progressively continuous. Let $h \in \mathcal{C}_t$. The upper semi-continuous envelop of $R(h)$ in viscosity sense

$$R(h)^*(s,\omega) = \lim_{\eta \to 0} \sup\{R(h)(s',\omega'),\ (s',\omega') \in D_\eta(s,\omega)\}$$

is a viscosity subsolution of the above path dependent second order PDE.